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Problem 10.13 (Zemansky)

Two identical bodies of constant heat capacity at the temperatures T_1 *and* T_2*, respectively, are used as reservoirs for a heat engine. If the bodies remain at constant pressure and undergo no change of phase, we have to show that the amount of work obtainable is*

$$mc_p\left(T_1+T_2-T_f\right),$$

where T_f *is the final temperature attained by both bodies. We have to show that, when W is a maximum,*

$$T_f=\sqrt{T_1T_2}.$$

Solution:

Let the mass of each of the two bodies be *m*. The temperature of the body 1 is T_1 and that of the body 2 is T_2. We assume that $T_1>T_2$. The two bodies are connected to an heat engine. Heat will be extracted from body 1 and part of which will be converted into work and part will be delivered to the body 2. This will go on till the temperature of both bodies becomes equal, say T_f. Then the heat engine will cease to function.

Let the total amount of heat extracted out by the heat engine from the body 1 be Q_1 and let the amount of heat delivered by the heat engine to the body 2 be Q_2. The amount of work done by the heat engine in the process described above be W. From the energy conservation, we have

$$W = Q_1 - Q_2.$$

As the initial temperature of the body 1 is T_1 and its final temperature is T_f, and its heat capacity at constant pressure per unit mass is c_p, we have

$$Q_1 = mc_p\left(T_1 - T_f\right).$$

Similarly the amount of heat received by the body 2 will be

$$Q_2 = mc_p\left(T_f - T_2\right).$$

Therefore, the amount of obtainable work from the two bodies will be

$$W = Q_1 - Q_2 = mc_p\left(T_1 + T_2 - 2T_f\right).$$

We next work out the change of entropy for the bodies 1 and 2. As the initial and final states of each body are well defined by their initial and final temperatures, we can compute change of entropy of each body assuming that the change takes place isobarically. We have

$$\Delta S_1 = mc_p \int_{T_1}^{T_f} \frac{dT}{T} = mc_p \ln\left(\frac{T_f}{T_1}\right).$$

And

$$\Delta S_2 = mc_p \int_{T_2}^{T_f} \frac{dT}{T} = mc_p \ln\left(\frac{T_f}{T_2}\right).$$

There is no change in entropy of the engine. Therefore, the total change of entropy is

$$\Delta S = \Delta S_1 + \Delta S_2 = mc_p \ln\left(\frac{T_f^{\,2}}{T_1 T_2}\right).$$

From the entropy principle, we have

$$\Delta S \geq 0,$$

or

$$mc_p \ln\left(\frac{T_f^2}{T_1 T_2}\right) \geq 0.$$

Maximum work will be obtainable for $\Delta S = 0$,

that is

$$\frac{T_f^2}{T_1 T_2} = 1,$$

or $\quad T_f = \sqrt{T_1 T_2}.$