927.

Problem 56.33 (RHK)

We have to show (a) *that the number N of photons radiated, per unit area per unit time, by a cavity radiator at temperature T is given by*

$$N = \int_0^\infty \frac{R(\lambda)}{hc/\lambda} d\lambda \approx \frac{30\sigma}{\pi^4 k} T^3.$$

In evaluating the integral, we will ignore "1" *in the denominator of* $R(\lambda)$. (b) *We have to show that to the same approximation, the fraction of photons by number, with energies greater than* 2.2 MeV *at a temperature of* 9×10^8 K *is* 2.1×10^{-10}.

Solution:

The radiated power per unit area that extends from λ to $\lambda + d\lambda$ is given by the Planck's blackbody radiation law,

$$R(\lambda) = \frac{2\pi c^2 h}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}.$$

The energy of a photon of wavelength λ is hc/λ.

Therefore, the number of photons having wavelength λ

emitted per unit area per second by a blackbody at temperature T will be given by

$$\frac{R(\lambda)d\lambda}{hc/\lambda},$$

and therefore, the number N of photons radiated, per unit area per unit time, by a cavity radiator at temperature T will be given by

$$N=\int_0^\infty \frac{R(\lambda)}{hc/\lambda}d\lambda=\int_0^\infty \frac{2\pi c^2 h}{\lambda^5}\times\frac{1}{e^{hc/\lambda kT}-1}\times\frac{\lambda}{hc}d\lambda.$$

In evaluating the integral, we will ignore "1" in the denominator of $R(\lambda)$. In this approximation

$$N=\int_0^\infty \frac{R(\lambda)}{hc/\lambda}d\lambda\approx\int_0^\infty \frac{2\pi c^2 h}{\lambda^5}\times\frac{1}{e^{hc/\lambda kT}}\times\frac{\lambda}{hc}d\lambda$$

$$=(2\pi c)\int_0^\infty \frac{e^{-hc/\lambda kT}}{\lambda^4}d\lambda.$$

For evaluating this integral, we make the change of variable

$$\frac{hc}{\lambda kT}=x,$$

and

$$-\frac{hc}{kT\lambda^2}d\lambda=dx.$$

$$N = (2\pi c)\left(\frac{kT}{hc}\right)^3 \int_0^\infty x^2 e^{-x} dx = (2\pi c)\left(\frac{kT}{hc}\right)^3 \times 2,$$

as, $\int_0^\infty x^2 e^{-x} dx = 2.$

Therefore,

$$N = \frac{4\pi k^3 T^3}{h^3 c^2}.$$

Stefan-Boltzmann constant

$$\sigma = \frac{2\pi^5 k^4}{15 c^2 h^3}.$$

We thus find that

$$N = \frac{4\pi k^3 T^3}{h^3 c^2} = \frac{4\pi k^3 \sigma}{h^3 c^2 \left(2\pi^5 k^4 / 15 h^3 c^2\right)} T^3 = \frac{30\sigma}{\pi^4 k} T^3.$$

(b)

We will show that to the same approximation, the fraction of photons by number, with energies greater than 2.2 MeV at a temperature of 9×10^8 K is 2.1×10^{-10}. Photons with energy greater than 2.2 MeV will have wavelength $\lambda \le \lambda_{2.2\ \text{MeV}}$,

$$\lambda_{2.2\ \text{MeV}} = \frac{hc}{2.2\ \text{MeV}} = \frac{4.14\times10^{-21}\ \text{Mev.s} \times\ 3\times10^8\ \text{m s}^{-1}}{2.2\ \text{MeV}}$$

$$= 5.65\times10^{-13}\ \text{m}.$$

Therefore,

$$x_{\min} = \frac{hc}{\lambda_{2.2\ \text{MeV}} kT} = \frac{6.63\times10^{-34}\ \text{J s}\times 3\times10^{8}\ \text{m s}^{-1}}{5.65\times10^{-13}\ \text{m}\times 1.380\times10^{-23}\ \text{J K}^{-1}\times 9\times10^{8}\ \text{K}}$$

$$= 2.83\times10 = 28.3\ .$$

And,

$$N_{\geq 2.2\ \text{MeV}} = (2\pi c)\left(\frac{kT}{hc}\right)^3 \int_{x_{\min}}^{\infty} x^2 e^{-x} dx$$

$$= (2\pi c)\left(\frac{kT}{hc}\right)^3 \left[e^{-x_{\min}}\left(x_{\min}{}^2 + 2x_{\min} + 2\right)\right]$$

$$= \frac{15\sigma}{\pi^4 k} T^3 \left[e^{-x_{\min}}\left(x_{\min}{}^2 + 2x_{\min} + 2\right)\right].$$

Therefore,

$$\frac{N_{\geq 2.2\ \text{MeV}}}{N} = \frac{1}{2}\times e^{-x_{\min}}\left(x_{\min}{}^2 + 2x_{\min} + 2\right).$$

And,

$$\frac{N_{\geq 2.2\ \text{MeV}}}{N} = \frac{1}{2}\times e^{-28.3}\left(28.3^2 + 2\times 28.3 + 2\right)$$

$$= 2.2\times10^{-10}.$$