925.

Problem 56.32 (RHK)

The existence of dark (*i.e. non-luminous*) *matter in a galaxy* (*such as our own*) *can be inferred by determining through observation the variation with distance in the orbital period of revolution of stars about the galactic centre. This is then compared with the variation derived on the basis of distribution of matter as is indicated by luminous material* (*mostly stars*). *Any significant deviation implies the existence of dark matter. For example, suppose that the matter* (*star, gas, dust*) *of a particular galaxy, total mass M, is distributed uniformly throughout a sphere of radius R. A star, mass m, is revolving about the centre of the galaxy in a circular orbit of radius* $r < R$. (a) *We have to show that the orbital speed v of the star is given by*

$$v = r\sqrt{GM/R^3},$$

and therefore that the period of T of revolution is given by

$$T = 2\pi\sqrt{R^3/GM},$$

independent of r. (b) *We have to find the corresponding formula for the orbital period assuming that the mass of the galaxy is strongly concentrated toward the centre of the galaxy, so that essentially all the mass is at distances less than r. These considerations applied to our own Milky Way galaxy indicate that substantial quantities of dark matter are present.*

Solution:

(a)

Suppose that the matter (star, gas, dust) of a particular galaxy, total mass *M*, is distributed uniformly throughout a sphere of radius *R*. The density of matter will be

$$\rho = \frac{M}{4\pi R^3/3}.$$

A star, mass *m*, is revolving about the centre of the galaxy in a circular orbit of radius $r < R$. The gravitational pull on the star will be due to the mass contained inside spherical volume of radius *r* centred at the centre of the galaxy. The equation of motion of the star, assuming it is moving with uniform speed in a circular orbit of radius *r*, will be

$$\frac{mv^2}{r} = \frac{Gm \times \left(\frac{4\pi r^3}{3}\right) \times \left(\frac{3M}{4\pi R^3}\right)}{r^2},$$

or

$$v^2 = \frac{GMr^2}{R^3},$$

or

$$v = r\sqrt{GM/R^3} \ .$$

Therefore. its period of revolution T is given by

$$T = \frac{2\pi r}{v} = \frac{2\pi r}{r\sqrt{GM/R^3}} = 2\pi\sqrt{R^3/GM} \ ,$$

which is independent of r.

(b)

We have to find the corresponding formula for the orbital period assuming that the mass of the galaxy is strongly concentrated toward the centre of the galaxy, so that essentially all the mass is at distances less than r. the equation of motion of the star, assuming it is moving with uniform speed v in a circular orbit of radius r, will be

$$\frac{mv^2}{r} = \frac{GmM}{r^2},$$

or

$$v = \sqrt{GM/r} \ .$$

And, its period of revolution will be

$$T = \frac{2\pi r}{v} = \frac{2\pi r}{\sqrt{GM/r}} = \frac{2\pi r^{3/2}}{\sqrt{GM}}.$$