923.

Problem 56.30 (RHK)

We have to answer whether the universe will continue to expand forever. For attacking this problem, we make the reasonable assumption that the recessional speed of v of a galaxy a distance r from us is determined only by the matter that lies inside a sphere of radius r centred on us. If the total mass inside this sphere is M, the escape speed $v_e = \sqrt{2GM/r}$. (a) *We have to show that the average density* ρ *inside the sphere must at least equal to the value given by*

$$\rho = 3H^2/8\pi G$$

to prevent unlimited expansion. (b) *We have to evaluate this* "*critical density*" *numerically*; *and have to express our answer in terms of* H-atoms/m^3. *Measurements of the actual density are difficult and complicated by the presence of the dark matter.*

Solution:

We make the reasonable assumption that the recessional speed of *v* of a galaxy a distance *r* from us is determined only by the matter that lies inside a sphere of radius r centred on us. The amount of mass contained inside a sphere of radius *r* and uniform density ρ is

$$M = \frac{4\pi r^3 \rho}{3}.$$

If the total mass inside this sphere is *M*, the escape speed

$$v_e = \sqrt{2GM/r}\ .$$

We thus have the following expression for the escape speed

$$v_e = \sqrt{2GM/r} = \left(\frac{2G \times \left(4\pi r^3 \rho/3\right)}{r} \right)^{1/2}$$

$$= \sqrt{\frac{8\pi G \rho r^2}{3}}.$$

According to the Hubble's law, the recessional speed of a galaxy at a distance r from us is

$v = Hr$, where *H* is the Hubble's constant. If the universe is to be prevented from unlimited expansion

$$v \le v_e\ .$$

Therefore, the condition for the density for a closed universe is given by

$$v = v_{\mathrm{e}},$$

or

$$Hr = \sqrt{\frac{8\pi G\rho r^2}{3}},$$

or

$$H^2 r^2 = \frac{8\pi G\rho r^2}{3},$$

or

$$\rho_c = \frac{3H^2}{8\pi G}.$$

(b)

We will evaluate this "critical density" numerically; and express our answer in terms of $\mathrm{H\text{-}atoms}/\mathrm{m}^3$.

$$\rho_c = 3H^2/8\pi G = \frac{3\times\left(67\ \mathrm{km\ s^{-1}}/\mathrm{Mpc}\right)^2}{8\pi\times 6.67\times 10^{-11}\,\mathrm{m^3/s^2.kg}}$$

$$= \frac{3\times\left(67\ \mathrm{km\ s^{-1}}/3.084\times 10^{19}\ \mathrm{km}\right)^2}{8\pi\times 6.67\times 10^{-11}\,\mathrm{m^3/s^2.kg}}$$

$$= 8.45\times 10^{-27}\ \mathrm{kg.\ m^{-3}}$$

$$= 8.45\times 10^{-27}\times\left(\frac{6.02\times 10^{23}}{1.00797\times 10^{-3}}\right)\ \mathrm{H\text{-}atoms\ m^{-3}}$$

$$= 5.04\ \mathrm{H\text{-}atoms\ m^{-3}}.$$