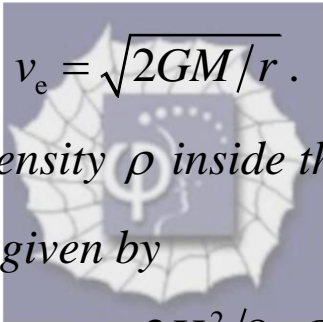


923.

Problem 56.30 (RHK)

We have to answer whether the universe will continue to expand forever. For attacking this problem, we make the reasonable assumption that the recessional speed of v of a galaxy a distance r from us is determined only by the matter that lies inside a sphere of radius r centred on us. If the total mass inside this sphere is M , the escape speed $v_e = \sqrt{2GM/r}$. (a) We have to show that the average density ρ inside the sphere must at least equal to the value given by


$$\rho = 3H^2/8\pi G$$

to prevent unlimited expansion. (b) We have to evaluate this “critical density” numerically; and have to express our answer in terms of H-atoms/m³. Measurements of the actual density are difficult and complicated by the presence of the dark matter.

Solution:

We make the reasonable assumption that the recessional speed of v of a galaxy a distance r from us is determined only by the matter that lies inside a sphere of radius r centred on us. The amount of mass contained inside a sphere of radius r and uniform density ρ is

$$M = \frac{4\pi r^3 \rho}{3}.$$

If the total mass inside this sphere is M , the escape speed

$$v_e = \sqrt{2GM/r}.$$

We thus have the following expression for the escape speed

$$\begin{aligned} v_e = \sqrt{2GM/r} &= \left(\frac{2G \times (4\pi r^3 \rho / 3)}{r} \right)^{1/2} \\ &= \sqrt{\frac{8\pi G \rho r^2}{3}}. \end{aligned}$$

According to the Hubble's law, the recessional speed of a galaxy at a distance r from us is

$v = Hr$, where H is the Hubble's constant. If the universe is to be prevented from unlimited expansion

$$v \leq v_e .$$

Therefore, the condition for the density for a closed universe is given by

$$v = v_e,$$

or

$$Hr = \sqrt{\frac{8\pi G \rho r^2}{3}},$$

or

$$H^2 r^2 = \frac{8\pi G \rho r^2}{3},$$

or

$$\rho_c = \frac{3H^2}{8\pi G}.$$



(b)

We will evaluate this “critical density” numerically; and express our answer in terms of H-atoms/m³.

$$\begin{aligned} \rho_c = 3H^2/8\pi G &= \frac{3 \times (67 \text{ km s}^{-1}/\text{Mpc})^2}{8\pi \times 6.67 \times 10^{-11} \text{ m}^3/\text{s}^2 \cdot \text{kg}} \\ &= \frac{3 \times (67 \text{ km s}^{-1}/3.084 \times 10^{19} \text{ km})^2}{8\pi \times 6.67 \times 10^{-11} \text{ m}^3/\text{s}^2 \cdot \text{kg}} \\ &= 8.45 \times 10^{-27} \text{ kg} \cdot \text{m}^{-3} \\ &= 8.45 \times 10^{-27} \times \left(\frac{6.02 \times 10^{23}}{1.00797 \times 10^{-3}} \right) \text{ H-atoms m}^{-3} \\ &= 5.04 \text{ H-atoms m}^{-3}. \end{aligned}$$

