801.

Problem 53.19 (RHK)

White dwarf stars represent a late stage in the evolution of stars like the Sun. They become dense enough and hot enough and that we can analyze their structure as a solid in which all Z electrons per atom are free. For a white dwarf with a mass equal to that of the Sun and a radius equal to that of the Earth, we have to calculate the Fermi energy of the electrons. We may assume that the atomic structure to be represented by iron atoms, and that $T = 0\ \text{K}$.

Solution:

We will calculate first the mass of an iron atom. The molar mass of iron is

$$M_{\text{Fe}} = 55.847 \text{ g mol}^{-1} = 55.847 \times 10^{-3} \text{kg mol}^{-1}.$$

Therefore, mass of an iron atom

$$m_{\text{Fe}} = \frac{M_{\text{Fe}}}{N_{\text{A}}} = \frac{55.847 \times 10^{-3}}{6.02 \times 10^{23}} \text{ kg}$$

$$= 9.277 \times 10^{-26} \text{ kg}.$$

Atomic number of iron $Z = 26$. Therefore, in its completely ionised state each iron atom contributes 26 free electrons.

We are given that the mass of the white dwarf star is one solar mass. That is

$$M_{\text{white dwarf}} = 1.99 \times 10^{30} \text{ kg}.$$

We are also given that the radius of the white dwarf star is equal to the radius of the earth. That is

$$R_{\text{white dwarf}} = 6.37 \times 10^{6} \text{ m}.$$

Therefore, the number of free electrons per cubic meter in the white dwarf star will be given by

$$n = \left(\frac{M_{\text{white dwarf}}}{m_{\text{Fe}}} \right) \times Z \times \frac{1}{\left(4\pi R^3_{\text{white dwarf}} / 3 \right)}$$

$$= \left(\frac{1.99 \times 10^{30}}{9.277 \times 10^{-26}} \right) \times 26 \times \frac{1}{\left(4\pi \left(6.37 \times 10^{6} \right)^3 / 3 \right)} \text{ m}^{-3}$$

$$= 5.15 \times 10^{35} \text{ m}^{-3}.$$

Fermi energy at absolute zero is the energy of the highest occupied state. It is a function of the number of free electrons per cubic meter, n, and is given by the expression

$$E_F = \frac{h^2}{8m} \left(\frac{3n}{\pi} \right)^{2/3}.$$

We use $n = 5.15\times10^{35}$ m^{-3} and find

$$
\begin{aligned}
E_F = \frac{h^2}{8m}\left(\frac{3n}{\pi}\right)^{2/3} &= \frac{\left(6.63\times10^{-34}\right)^2}{8\times9.11\times10^{-31}}\times\left(\frac{3\times5.15\times10^{35}}{\pi}\right)^{2/3}\ \text{J} \\
&= 37.56\times10^{-14}\ \text{J} \\
&= 37.56\times10^{-14}\times6.242\times10^{18}\ \text{eV} \\
&= 2.34\ \text{MeV}.
\end{aligned}
$$