766.

Problem 51.62 (RHK)

For a hydrogen atom in a state with $n = 2$ *and* $l = 0$, *we have to find the probability of finding the electron somewhere within the smaller of the two maxima of its radial probability density function.*

Solution:

The wave function for the state $n = 2$ and $l = 0$ is

$$\psi_{200}(r) = \frac{1}{4\sqrt{2\pi a^3}}(2 - r/a)e^{-r/2a} .$$

The radial probability density $P_r(r)$ is given by

$$P_r(r) = \psi^2_{200}(r)(4\pi r^2)$$

$$= \frac{1}{8a}\left(\frac{r}{a}\right)^2 (2 - r/a)^2 e^{-r/a}.$$

For finding out the maxima of $P_r(r)$, we calculate

$$\frac{dP_r(r)}{dr}.$$

We find

$$\frac{dP_r(r)}{dr} = \frac{1}{8a^2}\left(\frac{r}{a}\right)\left(2-\frac{r}{a}\right)e^{-r/a}\left\{4-\frac{6r}{a}+\left(\frac{r}{a}\right)^2\right\}.$$

It can be shown that $r = 2a$ is a minimum and the maxima of $P_r(r)$ are the roots of the equation

$$4-\frac{6r}{a}+\left(\frac{r}{a}\right)^2 = 0,$$

which are

$$\frac{r}{a} = 3-\sqrt{5},\ 3+\sqrt{5}\ .$$

The smaller of the maxima of the radial probability function is

$$a' = \left(3-\sqrt{5}\right)a = 0.76a\ .$$

The probability of finding the electron somewhere within the smaller of the two maxima of its radial probability density function will therefore be

$$
\begin{aligned}
P &= \int_0^{a'} \frac{1}{8a}\left(\frac{r}{a}\right)^2 (2 - r/a)^2 e^{-r/a} dr \\
&= \frac{1}{8}\int_0^{0.76} \xi^2 (2-\xi)^2 e^{-\xi} d\xi \\
&= \frac{1}{8}\int_0^{0.76} \left(4\xi^2 - 4\xi^3 + \xi^4\right) e^{-\xi} d\xi \\
&= 1 - e^{-0.76}\left(\frac{(0.76)^4}{8} + \frac{(0.76)^2}{2} + (0.76) + 1\right) \\
&= 1 - (0.47)(0.042 + 0.29 + 0.76 + 1) = 1 - 0.98 \\
&= 0.02\ .
\end{aligned}
$$

The probability is 2%.