**761.**

## Problem 40.50P (HRW)

*The wave function for the hydrogen atom quantum state, which has* $n = 2$, *and* $l = m_l = 0$, *is*

$$\psi_{200}(r) = \frac{1}{4\sqrt{2\pi}} a^{-3/2}\left(2 - \frac{r}{a}\right)e^{-r/2a},$$

*in which a is the Bohr radius and the subscript on* $\psi(r)$ *gives the values of the quantum numbers* $n,\ l,\ m_l$. (a) We have to show analytically that $\psi_{200}^2(r)$ has a maximum at $r = 4a$, *and* (b) *we have to find the radial probability density* $P_{200}(r)$ *for this state.* (c) *We have to show that*

$$\int_0^\infty P_{200}(r)dr = 1,$$

and thus that the expression above for the wave function $\psi_{200}(r)$ *has been properly normalized.*

## Solution:

(a)

The wave function for the hydrogen atom quantum state, which has $n = 2$, and $l = m_l = 0$, is

$$\psi_{200}(r) = \frac{1}{4\sqrt{2\pi}} a^{-3/2} \left(2 - \frac{r}{a}\right) e^{-r/2a}.$$

The square of the wave function $\psi_{200}(r)$ will be

$$\psi_{200}^2(r) = \frac{1}{32\pi a^3} \left(2 - \frac{r}{a}\right)^2 e^{-r/a}.$$

We will calculate next $\frac{d\psi_{200}^2(r)}{dr}$.

The condition

$\frac{d\psi_{200}^2(r)}{dr} = 0$ will determine the extremum of the

function $\psi_{200}^2(r)$. We have

$$\frac{d\psi_{200}^2}{dr} = \frac{1}{32\pi a^3} \left[ 2\left(2 - \frac{r}{a}\right)\left(-\frac{1}{a}\right) e^{-r/a} + \left(2 - \frac{r}{a}\right)^2 \left(-\frac{1}{a}\right) e^{-r/a} \right]$$

$$= -\frac{e^{-r/a}}{32\pi a^4} \left(2 - \frac{r}{a}\right) \left[2 + 2 - \frac{r}{a}\right].$$

The solutions of the equation $\frac{d\psi_{200}^2(r)}{dr} = 0$ are

$r = 2a$, and $r = 4a$.

One can show that $r = 2a$ is a minimum and $r = 4a$ is a maximum.

(b)

The radial probability density $P_{200}(r)$ will be

$4\pi r^2 \psi_{200}^2(r)$ and its expression will be

$$P_{200}(r) = \frac{1}{8a^3} r^2 \left(2 - \frac{r}{a}\right)^2 e^{-r/a}.$$

(c)

We next check the normalization of the wave function

$\psi_{200}(r)$ by calculating the integral

$$\int_0^\infty P_{200}(r) dr.$$

We have

$$\int_0^\infty P_{200}(r) dr = \int_0^\infty \frac{1}{8a^3} r^2 \left(2 - \frac{r}{a}\right)^2 e^{-r/a} dr$$
$$= \frac{1}{8a^3} \int_0^\infty dr \left(4r^2 + \frac{r^4}{a^2} - \frac{4r^3}{a}\right) e^{-r/a}.$$

We evaluate the integral by making the following

substitutions:

$\frac{r}{a} = \xi$, and $dr = a d\xi$.

We get

$$\int_0^\infty P_{200}(r)dr = \frac{1}{8}\int_0^\infty d\xi e^{-\xi}\left(4\xi^2 - 4\xi^3 + \xi^4\right)$$

$$= \frac{1}{8}\left(4\Gamma(3) - 4\Gamma(4) + \Gamma(5)\right)$$

$$= \frac{1}{8}\left(4\times 2! - 4\times 3! + 4!\right) = \frac{1}{8}(8) = 1.$$