755.

Problem 51.35 (RHK)

Of the three scalar components of $\vec{L}$*, one* L_z*, is quantized. We have to show that the most that can be said about the other two components of* $\vec{L}$ *is*

$$\sqrt{L_x^2 + L_y^2} = \sqrt{l\left(l+1\right) - m_l^2}\,\hbar$$

We note that these two components are not separately quantized. We have to show that

$$\sqrt{l}\,\hbar \le \sqrt{L_x^2 + L_y^2} \le \sqrt{l\left(l+1\right)}\,\hbar\ .$$

Solution:

In quantum mechanics it is shown that $\vec{L}.\vec{L} = L_x^2 + L_y^2 + L_z^2$ and one of the components, say, L_z can be simultaneously measured and that their values are given by

$$\vec{L}.\vec{L} = L_x^2 + L_y^2 + L_z^2 = l\left(l+1\right)\hbar^2,\ l = 0,1,2,3..$$

and for a given value of l the allowed values of L_z are

$$m_l\hbar,\ m_l = l, l-1, l-2, .., 0, 1, ..., -l+1, -l.$$

Combining these two results, we note that what can be said about $L_x^2 + L_y^2$ is that as

$$L_x^2 + L_y^2 = \vec{L}\cdot\vec{L} - L_z^2,$$

its values can be

$$L_x^2 + L_y^2 = l\left(l+1\right)\hbar^2 - m_l^2\hbar^2,$$

or

$$\sqrt{L_x^2 + L_y^2} = \sqrt{l\left(l+1\right)\hbar^2 - m_l^2\hbar^2}$$

$$= \sqrt{l\left(l+1\right) - m_l^2}\,\hbar$$

As the maximum value of $\left|m_l\right| = l$,

we note that

$$\sqrt{L_x^2 + L_y^2} \geq \sqrt{l\left(l+1\right) - l^2}\,\hbar,$$

or

$$l\hbar \leq \sqrt{L_x^2 + L_y^2} \; .$$

As the least value of $\left|m_l\right| = 0$, we note that

$$\sqrt{L_x^2 + L_y^2} \; \leq \sqrt{l\left(l+1\right)}\hbar \; .$$

We thus find that

$$\sqrt{l}\hbar \leq \sqrt{L_x^2 + L_y^2} \leq \sqrt{l\left(l+1\right)}\hbar.$$