709.

Problem 49.18 (RHK)

Given that an ideal radiator has a spectral radiancy at 400 nm *that is* 3.50 *times its spectral radiancy at* 200 nm. (a) *We have to find the temperature of the radiator*. (b) *We have to calculate the temperature if the spectral radiancy of the radiator at* 200 nm were 3.5 *times its spectral radiancy at* 400 nm.

Solution:

According to Planck's radiation law the spectral radiancy of a radiator at temperature T and wavelength λ is given by the following function:

$$R(\lambda) = \frac{2\pi c^2 h}{\lambda^5} \times \frac{1}{\exp(hc/\lambda kT) - 1},$$

where

the Boltzmann constant $k = 1.381 \times 10^{-23}$ J K^{-1}

and the Planck constant $h = 6.626 \times 10^{-34}$ J s

(a)

It is given that at temperature T the radiancy at $\lambda = 200 \text{ nm}$ and at $\lambda' = 2\lambda = 400 \text{ nm}$ are related as follows:

$$R(\lambda' = 400 \text{ nm}) = 3.5 \times R(\lambda = 200 \text{ nm}).$$

We therefore have the equation

$$\frac{1}{(\lambda')^5} \times \frac{1}{\exp(hc/\lambda' kT) - 1} = 3.5 \times \frac{1}{\lambda^5} \frac{1}{\exp(hc/\lambda kT) - 1}$$

or

$$\frac{1}{2^5} \times \frac{1}{x-1} = 3.5 \times \frac{1}{x^2 - 1},$$

where

$$x = \exp(hc/2\lambda kT).$$

The above equation readily simplifies to the form

$$x^2 - 112x + 111 = 0,$$

or

$$(x - 1)(x - 111) = 0.$$

Root $x = 1$ is unphysical. We find temperature T corresponding to the root

$$x = 111,$$

we have

$$\frac{hc}{2\lambda kT} = \ln(111),$$

or

$$T = \frac{6.626 \times 10^{-34} \times 3 \times 10^{8}}{400 \times 10^{-9} \times 1.381 \times 10^{-23} \times \ln(111)} \text{ K}$$

$$= 7641 \text{ K}.$$

(b)

We next solve the second part of the problem.

The condition is

$$R(200 \text{ nm}) = 3.5 \times R(400 \text{ nm}).$$

We once again make the substitution

$$x = \exp(hc/2\lambda kT).$$

For this case, we solve the following equation for finding the temperature of the radiator:

$$\frac{1}{x^2 - 1} = \frac{3.5}{2^5} \times \frac{1}{x - 1},$$

or

$$(x - 1)\left(x - \left(\frac{2^5}{3.5} - 1\right)\right) = 0.$$

We select the physical root

$$x = \left(\frac{2^5}{3.5} - 1 \right) = 8.143.$$

We next solve

$$\exp\left(hc/2\lambda kT \right) = 8.143,$$

for

$$\lambda = 200 \text{ nm} = 2.0 \times 10^{-7} \text{ m},$$

and find that

$$T = 17{,}170 \text{ K}.$$