536.

Problem 39.21 (RHK)

We have to show that the fractional width of the resonance curves in LCR circuits is given, to a close approximation, by

$$\frac{\Delta\omega}{\omega_R} = \frac{\sqrt{3}R}{\omega_R L} \ ,$$

in which ω_R *is the resonant frequency and* $\Delta\omega$ *is the width of the resonance peak at* $i = i_m/2$. *Note that this expression may be written as* $\sqrt{3}/Q$, *which shows that clearly that a "high-Q" circuit has a sharp resonance peak, that is, a small value of* $\Delta\omega/\omega_R$.

Solution:

Let an *LCR* circuit be driven by a sinusoidal emf

$\mathrm{E} = \mathrm{E}_m \sin\omega t$,

and let the current in the circuit be given by the function

$i = i_m \sin(\omega t - \phi)$.

In an *LCR* circuit the current amplitude and the emf amplitude are related to each other by the impedance

$$Z = \sqrt{R^2 + \left(L\omega - 1/C\omega\right)^2},$$

as

$$i_m = \frac{\mathcal{E}_m}{\sqrt{R^2 + \left(L\omega - 1/C\omega\right)^2}}.$$

The peak value of i_m is $\mathcal{E}_m/R$ at the resonance frequency

$$\omega_R = \frac{1}{\sqrt{LC}} .$$

Let at frequencies ω_{d_1} and ω_{d_2} the peak current be half of i_m, that is $\mathcal{E}_m/2R$.

We, therefore, have the equation from which the frequencies ω_d can be calculated:

$$\frac{\mathcal{E}_m}{2R} = \frac{\mathcal{E}_m}{\sqrt{R^2 + \left(L\omega_d - 1/C\omega_d\right)^2}},$$

or

$$R^2 + \left(L\omega_d - 1/C\omega_d\right)^2 = 4R^2,$$

or

$$\left(L\omega_d - 1/C\omega_d\right) = \pm\sqrt{3}R.$$

We assume

$\omega_d = \omega_R + \Delta\omega$, and $\Delta\omega/\omega_R = 1$.

We thus have the equation

$$\omega_R L + \Delta\omega L - \frac{1}{C\left(\omega_R + \Delta\omega\right)} = \pm\sqrt{3}R,$$

or

$$\omega_R L + \Delta\omega L - \frac{1}{C\omega_R}\left(1 - \frac{\Delta\omega}{\omega_R} + O\left(\frac{\Delta\omega}{\omega_R}\right)^2\right) = \pm\sqrt{3}R,$$

or

$$\omega_R L + 2\Delta\omega L - \frac{1}{C\omega_R} = \pm\sqrt{3}R,$$

or

$$\Delta\omega = \frac{\pm\sqrt{3}R}{2L}.$$

Therefore,

$$\omega_{d_1} = \omega_R - \frac{\sqrt{3}R}{2L}, \text{ and } \omega_{d_2} = \omega_R + \frac{\sqrt{3}R}{2L}.$$

Therefore, the width of the resonance peak at half maximum is

$$\frac{\omega_{d_2} - \omega_{d_1}}{\omega} = \frac{\sqrt{3}R}{L\omega} = \frac{\sqrt{3}}{Q},$$

where “quality” of an *LCR* is defined as

$$Q = \frac{\omega L}{R}.$$