

536.

**Problem 39.21 (RHK)**

We have to show that the fractional width of the resonance curves in LCR circuits is given, to a close approximation, by

$$\frac{\Delta\omega}{\omega_R} = \frac{\sqrt{3}R}{\omega_R L},$$

in which  $\omega_R$  is the resonant frequency and  $\Delta\omega$  is the width of the resonance peak at  $i = i_m/2$ . Note that this expression may be written as  $\sqrt{3}/Q$ , which shows that clearly that a “high- $Q$ ” circuit has a sharp resonance peak, that is, a small value of  $\Delta\omega/\omega_R$ .



**Solution:**

Let an LCR circuit be driven by a sinusoidal emf

$$E = E_m \sin \omega t,$$

and let the current in the circuit be given by the function

$$i = i_m \sin(\omega t - \phi).$$

In an LCR circuit the current amplitude and the emf amplitude are related to each other by the impedance

$$Z = \sqrt{R^2 + (L\omega - 1/C\omega)^2},$$

as

$$i_m = \frac{E_m}{\sqrt{R^2 + (L\omega - 1/C\omega)^2}}.$$

The peak value of  $i_m$  is  $E_m/R$  at the resonance frequency

$$\omega_R = \frac{1}{\sqrt{LC}}.$$

Let at frequencies  $\omega_{d_1}$  and  $\omega_{d_2}$  the peak current be half of  $i_m$ , that is  $E_m/2R$ .

We, therefore, have the equation from which the frequencies  $\omega_d$  can be calculated:

$$\frac{E_m}{2R} = \frac{E_m}{\sqrt{R^2 + (L\omega_d - 1/C\omega_d)^2}},$$

or

$$R^2 + (L\omega_d - 1/C\omega_d)^2 = 4R^2,$$

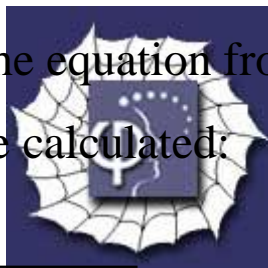
or

$$(L\omega_d - 1/C\omega_d) = \pm\sqrt{3}R.$$

We assume

$$\omega_d = \omega_R + \Delta\omega, \text{ and } \Delta\omega/\omega_R = 1.$$

We thus have the equation



$$\omega_R L + \Delta\omega L - \frac{1}{C(\omega_R + \Delta\omega)} = \pm\sqrt{3}R,$$

or

$$\omega_R L + \Delta\omega L - \frac{1}{C\omega_R} \left( 1 - \frac{\Delta\omega}{\omega_R} + O\left(\frac{\Delta\omega}{\omega_R}\right)^2 \right) = \pm\sqrt{3}R,$$

or

$$\omega_R L + 2\Delta\omega L - \frac{1}{C\omega_R} = \pm\sqrt{3}R,$$

or

$$\Delta\omega = \frac{\pm\sqrt{3}R}{2L}.$$

Therefore,

$$\omega_{d_1} = \omega_R - \frac{\sqrt{3}R}{2L}, \text{ and } \omega_{d_2} = \omega_R + \frac{\sqrt{3}R}{2L}.$$



Therefore, the width of the resonance peak at half maximum is

$$\frac{\omega_{d_2} - \omega_{d_1}}{\omega} = \frac{\sqrt{3}R}{L\omega} = \frac{\sqrt{3}}{Q},$$

where “quality” of an *LCR* is defined as

$$Q = \frac{\omega L}{R}.$$