531.

Problem 38.71 (RHK)

We have to show that in a damped LC circuit the fraction of the energy lost per cycle of oscillation, $\Delta U/U$, *is given to a close approximation by* $2\pi R/\omega L$. *The quantity* $\omega L/R$ *is often called the Q of the circuit (for "quality"). A "high-Q" circuit has low resistance and a low fractional energy loss per cycle* $(=2\pi/Q)$.

Solution:

The differential equation giving the variation of charge q as a function of time in a damped LC circuit is

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=0.$$

The solution of this differential equation is

$$q=q_m e^{-Rt/2L}\cos(\omega' t+\phi),$$

where

$$\omega'=\sqrt{\omega^2-(R/2L)^2}\ .$$

Assuming that

$$\omega = \frac{1}{\sqrt{LC}} \ ? \ \frac{R}{2L},$$

current can be approximated by the function

$$i = \frac{dq}{dt} \ ; \ -q_m \omega e^{-Rt/2L} \sin\left(\omega' t + \phi\right).$$

The total electromagnetic energy at any instant in the *LC* circuit is the sum of electrical energy stored in the capacitor and the magnetic energy stored in the inductor. That is

$$U = \frac{1}{2} L i^2 + \frac{q^2}{2C} = \frac{1}{2} L\omega^2 \left(q_m\right)^2 e^{-Rt/L} \sin^2\left(\omega' t + \phi\right) +$$

$$+ \frac{1}{2C}\left(q_m\right)^2 e^{-Rt/L} \cos^2\left(\omega' t + \phi\right)$$

$$= \frac{1}{2} L\omega^2 \left(q_m\right)^2 e^{-Rt/L}.$$

The dissipation of energy in the *LC* circuit in the resistor *R* is given by the relation

$$\frac{dU}{dt} = -i^2 R \ ; \ -\left(q_m\right)^2 R\omega^2 e^{-Rt/L} \sin^2\left(\omega' t + \phi\right).$$

The total dissipation of energy over one cycle will therefore be given by the integral

$$\Delta U = \int_0^{2\pi/\omega} \frac{dU}{dt} dt = q_m^2 \omega^2 R \, \mathrm{e}^{-Rt/L} \times \frac{1}{2} \times \frac{2\pi}{\omega}$$

$$= \pi q_m^2 \omega R \, \mathrm{e}^{-Rt/L}.$$

$$\therefore \quad \frac{\Delta U}{U} = \frac{2\pi R}{L\omega}.$$

The quantity $\omega L/R$ is called the Q of the circuit.

Thus

$$\frac{\Delta U}{U} = \frac{2\pi}{Q}.$$