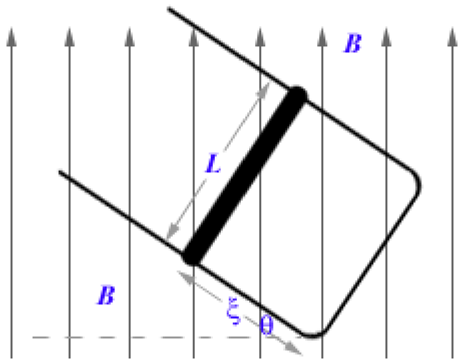
493.

Problem 36.37 (RHK)

A rod with length L, mass m, and resistance R slides without friction down parallel conducting rails of negligible resistance, as shown in the figure. The rails are connected together at the bottom as shown, forming a conducting loop with the rod as the top member. The plane of the rails makes an angle θ with the horizontal and a uniform vertical magnetic field $\vec{B}$ exists throughout the region. (a) *We have to show that the rod acquires a steady-state terminal velocity whose magnitude is*

$$v = \frac{mgR}{B^2L^2}\frac{\sin\theta}{\cos^2\theta}.$$

(b) *We have to show that the rate at which internal energy is being generated in the rod is equal to the rate at which the rod is losing gravitational potential energy.*
(c) *We have to discuss the situation if $\vec{B}$ were directed down instead of up.*



Solution:

A rod with length *L*, mass *m*, and resistance *R* slides without friction down parallel conducting rails of negligible resistance, as shown in the figure. The rails are connected together at the bottom as shown, forming a conducting loop with the rod as the top member. The plane of the rails makes an angle θ with the horizontal and a uniform vertical magnetic field $\vec{B}$ exists throughout the region.

(a)

Let at a particular instant the distance of the rod from the bottom end of the rails be ξ. The flux enclosed by the loop spanned by the rod, the rails and the bottom connector will be

$\Phi = L\xi B\cos\theta,$

As the angle between the magnetic field $\vec{B}$ and the normal to the plane of the loop is θ. When the rod begins to slide down the rails, the flux enclosed will change and an induced emf will be produced in the circuit.

By Faraday’s law of induction, the magnitude of the induced emf will be

$$\mathrm{E} = \frac{d\Phi}{dt} = LB\cos\theta \frac{d\xi}{dt}.$$

An induced current will flow in the loop in the counter-clockwise direction as seen from the top to resist the change in flux. As the resistance of the conducting rod is *R*, the magnitude of the current will be

$$i = \frac{LB\cos\theta}{R}\frac{d\xi}{dt}.$$

Because of the magnetic field $\vec{B}$ the conducting rod will experience a Lorentz force. Its component in the plane spanned by the rod and the rails will be

$$F_{induction} = \left(\frac{LB\cos\theta}{R}\frac{d\xi}{dt}\right)LB\cos\theta.$$

This force will oppose the sliding force due to gravity of magnitude

$$F_{gravity} = mg\sin\theta.$$

Let the terminal speed of the rod be *v*. That is then

$$\frac{d\xi}{dt} = v.$$

We, therefore, have the condition

$$F_{induction} = F_{gravity}$$

or

$$\frac{L^2 B^2 \cos^2\theta}{R} v = mg\sin\theta,$$

and

$$v = \frac{mgR\sin\theta}{L^2 B^2 \cos^2\theta}.$$

(b)

We will next prove that this result is consistent with the energy conservation principle. When the rod begins to slide with constant speed v, the loss in gravitational potential energy should be equal to the Joule heat in the conductor.

In time interval *t* the rod will slide by distance *vt* (we measure the time from the instant when the rod attains the terminal speed *v*). During this time interval the change in the potential energy of the conducting rod will be

$$\Delta U_g = mgvt\sin\theta.$$

The internal energy dissipated as Joule heat during the time interval t will be

$$\Delta U_J = \mathrm{E}it = \frac{\left(LB\cos\theta v\right)^2}{R}t.$$

We recall that

$$\frac{L^2B^2\cos^2\theta}{R} = \frac{mg\sin\theta}{v}.$$

Therefore, we note that

$$\Delta U_g = \Delta U_J,$$

as required by the energy conservation principle.

If the magnetic field were directed down instead of up, the situation would remain essentially unchanged, except the direction of the induced current would now be clockwise as seen from the top.