463.

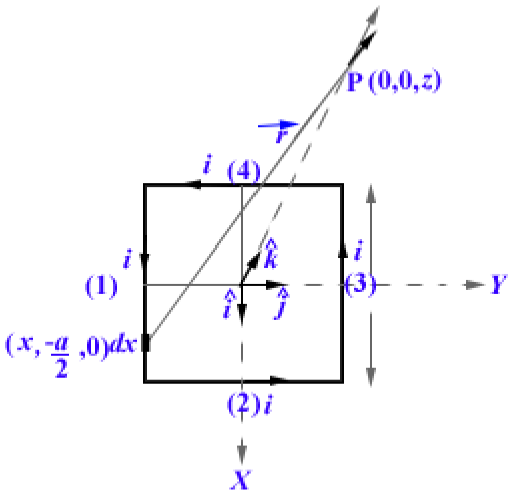
Problem 35.18 (RHK)

A *square loop of wire of edge a carries a current* i.
(a) *We have to show that B for a point on the axis of the loop and a distance z from its centre is given by*

$$B(z)=\frac{4\mu_0 i a^2}{\pi\left(4z^2+a^2\right)\left(4z^2+2a^2\right)^{1/2}}.$$

(b) *From this result we have to answer, what does this field reduce to at the centre of the square?*

Solution:



We consider a square loop of side *a* in which a current i is flowing in the counter-clockwise direction. The loop is in the *xy*-plane, as shown in the figure. Z-axis is perpendicular to the plane and passes through the centre of the square. We number the four segments of the loop as shown in the figure.

We will first calculate the contribution to the magnetic field $\vec{B}$ at point P $(0,0,z)$ on the axis of the loop due to the segment (1).

Let us consider an infinitesimal current element $i\vec{dx}$ along the direction of the current at point $\left(x,-\frac{a}{2},0\right)$. Let the line vector joining P with dx be $\vec{r}$. Length of $\vec{r}$ will be

$$r=\left(x^2+\frac{a^2}{4}+z^2\right)^{1/2}$$

and the unit vector in the direction of $\vec{r}$ is

$$\hat{r}=-\frac{x}{r}\hat{i}+\frac{a}{2r}\hat{j}+\frac{z}{r}\hat{k}.$$

The magnetic field at P due to $i\vec{dx}$ will be

$$d\vec{B}_{(1)}=\frac{\mu_0}{4\pi}\frac{i\vec{dx}\times\hat{r}}{r^2}=\frac{\mu_0 idx}{4\pi r^2}\left(\hat{i}\times\left(-\frac{x}{r}\hat{i}+\frac{a}{2r}\hat{j}+\frac{z}{r}\hat{k}\right)\right)$$

$$=\frac{\mu_0 idx}{4\pi r^2}\left(\frac{a}{2r}\hat{k}-\frac{z}{r}\hat{j}\right).$$

The component of $d\vec{B}_{(1)}$ in direction of $\hat{k}$ is

$$dB_{(1)} = \frac{\mu_0}{4\pi}\frac{iadx}{2r^3} = \frac{\mu_0}{4\pi}\frac{iadx}{2\left(x^2 + \frac{a^2}{4} + z^2\right)^{3/2}}.$$

We may note that by symmetry the net contribution in the $\hat{j}$ direction will be zero. Therefore,

$$B_{(1)} = 2\int_0^{a/2} \frac{\mu_0 ai}{8\pi}\frac{dx}{\left(x^2 + \frac{a^2}{4} + z^2\right)^{3/2}} = \frac{\mu_0 ia^2}{\pi\left(a^2 + 4z^2\right)\left(2a^2 + 4z^2\right)^{1/2}}.$$

Contributions to the magnetic field at P due to the other three segments will be equal to the one calculated for the segment (1) and will also be in the $\hat{k}$ direction. Therefore, the magnetic field at P due to the square loop will be

$$B(z) = \frac{4\mu_0 ia^2}{\pi\left(4z^2 + a^2\right)\left(4z^2 + 2a^2\right)^{1/2}}.$$

(b)

Therefore, the field at the centre of the square will be

$$B(0,0,0) = \frac{2\sqrt{2}\mu_0 i}{\pi a}.$$