424.

Problem 33.55 (RHK)

A 3.0-$\text{M}\Omega$ *resistor and a* 1.0-μF *capacitor are connected in a single loop circuit with a seat of emf with* $\mathcal{E} = 4.0\text{ V}$. *At* 1.0 s *after the connection is made, we have to find the rates at which* (a) *the charge on the capacitor is increasing,* (b) *the energy is being stored in the capacitor,* (c) *internal energy is appearing in the resistor, and* (d) *the energy is being delivered by the seat of emf.*

Solution:

The variation of charge on a capacitor with time during the charging process is given by the equation

$$q(t) = q_0\left(1 - e^{-t/RC}\right),$$

where, $q_0 = \mathcal{E}C.$

The rate of change of charge, $\frac{dq(t)}{dt}$, is the current $i(t)$.

It is

$$i(t) = \frac{dq(t)}{dt} = \frac{q_0}{RC}e^{-t/RC}. \quad (\text{A})$$

The internal energy in the capacitor at time, *t*, will be

$$U_C(t) = \frac{q^2(t)}{2C}.$$

Therefore,

$$\frac{dU_C(t)}{dt} = \frac{q(t)}{C} \times \frac{dq(t)}{dt} = \frac{q_0}{C}\left(1 - e^{-t/RC}\right) \times \frac{q_0}{RC} e^{-t/RC}. \ (\mathrm{B})$$

The rate at which internal energy is appearing in the resistor is

$$P_R(t) = i^2(t) R = \left(\frac{q_0 e^{-t/RC}}{RC}\right)^2 \times R = \frac{q_0^2}{RC^2} e^{-2t/RC}. \ (\mathrm{C})$$

The rate at which the energy is being delivered by the seat of emf is

$$P_B(t) = \mathrm{E}\, i(t) = \frac{\mathrm{E} q_0}{RC} e^{-t/RC}. \ (\mathrm{D})$$

Data of our problem are:

$R = 3.0 \times 10^6\ \Omega$,

$C = 1.0 \times 10^{-6}\ \mathrm{F}$,

$\therefore\ RC = 3\ \mathrm{s}$,

$q_0 = \mathrm{E} C = 4.0 \times 10^{-6}\ \mathrm{C}$.

In equations (A), (B), (C) and (D), we will substitute the above values and use $t = 1.0\ \mathrm{s}$. We find

(a)

The rate of change of charge at $t = 1.0\ \text{s}$ is

$$\frac{dq(t = 1.0\ \text{s})}{dt} = \frac{4.0\times10^{-6}}{3}e^{-1/3} = 0.955\ \mu\text{C s}^{-1},$$

as

$$e^{-1/3} = 0.7165.$$

(b)

The rate at which the energy is being stored in the capacitor is at $t = 1.0\ \text{s}$ is

$$\begin{aligned}\frac{dU_C(t = 1.0\ \text{s})}{dt} &= \frac{\mathcal{E}^2}{R}\left(1 - e^{-1/3}\right)e^{-1/3} \\ &= \frac{16}{3.0\times10^{6}}\left(1 - e^{-1/3}\right)e^{-1/3}\ \text{J s}^{-1} \\ &= 1.08\ \mu\text{W}.\end{aligned}$$

(c)

The rate at which internal energy is appearing in the resistor at $t = 1.0\ \text{s}$ is

$$\begin{aligned}P_R(t = 1.0\ \text{s}) = \frac{q_0^2}{RC^2}e^{-2/3} = \frac{\mathcal{E}^2}{R}e^{-2/3} &= \frac{16}{3\times10^{6}}e^{-2/3}\ \text{W} \\ &= 2.74\ \mu\text{W}.\end{aligned}$$

(d)

The rate at which the energy is being delivered by the seat of emf at $t = 1.0\ \text{s}$ is

$$P_B\left(t=1.0\text{ s}\right)=\frac{\mathrm{E}q_0}{RC}e^{-1/3}=\frac{\mathrm{E}^2}{R}e^{-1/3}=\frac{16}{3\times10^6}e^{-1/3}\text{ W}$$

$$=3.82\ \mu\text{W}.$$