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Problem 33.58 (RHK)

An initially uncharged capacitor C is fully charged by a constant emf $\mathcal{E}$ *in series with a capacitor R.* (a) *We have to show that the final energy stored in the capacitor is half the energy supplied by the emf.* (b) *By direct integration of* i^2R *over the charging time, we have to show that the internal energy dissipated by the resistor is also half the energy supplied by the emf.*

Solution:

(a)

The charging equation for a capacitor of capacitance C connected with a resistor or resistance *R* to a source of emf $\mathcal{E}$ is

$$q(t) = q_0\left(1 - e^{-t/RC}\right),$$

where

$$q_0 = C\mathcal{E}.$$

By differentiating $q(t)$ with respect to the variable *t*, we will obtain the current as a function of time, $i(t)$, during the charging process. That is

$$i(t)=\frac{dq(t)}{dt}=-q_0\left(-\frac{1}{RC}\right)e^{-t/RC}=\frac{q_0}{RC}e^{-t/RC}=\frac{\mathrm{E}}{R}e^{-t/RC}.$$

The energy supplied by the emf in charging the capacitor fully will be given by the integral

$$U=\int_0^\infty \mathrm{E}\,i(t)dt=\int_0^\infty \frac{\mathrm{E}^2}{R}e^{-t/RC}dt=\frac{\mathrm{E}^2}{R}(-RC)\left[e^{-t/RC}\right]_0^\infty$$

$$=C\mathrm{E}^2.$$

The final energy stored in the capacitor is

$$U_C=\frac{q_0^2}{2C}=\frac{(C\mathrm{E})^2}{2C}=\frac{C\mathrm{E}^2}{2}.$$

Therefore, it is half the energy supplied by the source of emf.

(b)

We next calculate the internal energy dissipated in the resistor.

Joule heat per second is

$$P(t)=i^2(t)R.$$

Therefore, the total energy dissipated in the resistor during the charging process will be given by integrating $P(t)$ over the charging time. That is

$$U_R = \int_0^\infty P(t)dt = \int_0^\infty i^2(t) R\, dt = \int_0^\infty \frac{\mathrm{E}^2}{R^2} e^{-2t/RC} R dt$$

$$= \frac{\mathrm{E}^2}{R} \int_0^\infty e^{-2t/RC} dt$$

$$= \frac{\mathrm{E}^2}{R}\left(-\frac{RC}{2}\right)\left[e^{-2t/RC}\right]_0^\infty$$

$$= \frac{\mathrm{E}^2 C}{2}.$$

This is also equal to half the energy supplied by the emf in charging the capacitor.