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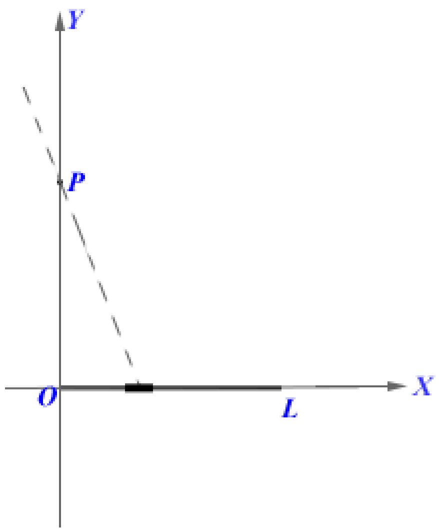
Problem 30.51 (RHK)

On a thin rod of length L lying on x axis with one end at the origin $(x=0)$, *as shown in the figure, there is distributed a charge per unit length given by* $\lambda = kx$, *where k is a constant.* (a) *Taking the electrostatic potential at infinity to be zero, we have to find V at point P on the y axis.* (b) *We have to determine the vertical component,* E_y, *of the electric field at P from the result of part* (a) *and also by direct calculation.* (c) *We have to answer why we cannot find* E_x, *the horizontal component of the electric field at P using the result of part* (a). (d) *We have to find the distance from the rod along the y axis where the potential is equal to one-half of its value at the left end of the rod.*



Solution:

(a)

Potential at P due to the charge on the rod can be calculated by integrating contribution from infinitesimal element of length dx,

$\dfrac{\lambda dx}{4\pi\varepsilon_0\left(x^2+y^2\right)^{1/2}}$. This definition of the potential ensures that the potential at ∞ is zero.

$$V(0,y)=\frac{1}{4\pi\varepsilon_0}\int_0^L\frac{\lambda dx}{\left(x^2+y^2\right)^{1/2}}=\frac{1}{4\pi\varepsilon_0}\int_0^L\frac{k\,x\,dx}{\left(x^2+y^2\right)^{1/2}}\ .$$

For performing the integration, we make the substitution

$$x^2+y^2=\xi,$$
$$2xdx=d\xi.$$

Therefore,

$$V(0,y)=\frac{1}{4\pi\varepsilon_0}\int_0^L \frac{k\,x\,dx}{\left(x^2+y^2\right)^{1/2}}=\frac{1}{4\pi\varepsilon_0}\int_{y^2}^{L^2+y^2}\frac{kd\xi}{2\xi^{1/2}}$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{k}{2}\left[\frac{\xi^{1/2}}{1/2}\right]_{y^2}^{L^2+y^2}$$

$$=\frac{k}{4\pi\varepsilon_0}\left(\left(L^2+y^2\right)^{1/2}-y\right).$$

The *y*-component of the electric field at *P* can be obtained by differentiating $V(0,y)$ with respect to y. that is

$$E_y=-\frac{dV(0,y)}{dy}=\frac{k}{4\pi\varepsilon_0}\left(1-\frac{y}{\left(L^2+y^2\right)^{1/2}}\right).$$

(b)

We will next calculate E_y directly

$$E_y(0,y)=\frac{1}{4\pi\varepsilon_0}\int_0^L \frac{kxdx}{\left(x^2+y^2\right)}\times\frac{y}{\left(x^2+y^2\right)^{1/2}}$$

$$=\frac{ky}{4\pi\varepsilon_0}\int_0^L \frac{xdx}{\left(x^2+y^2\right)^{3/2}}.$$

For performing the integration, we once again make the substitution

$$x^2 + y^2 = \xi,$$
$$2xdx = d\xi.$$

We get

$$E_y\left(0, y\right) = \frac{ky}{4\pi\varepsilon_0} \int_{y^2}^{L^2+y^2} \frac{d\xi}{2\xi^{3/2}}$$

$$= \frac{ky}{4\pi\varepsilon_0} \times \frac{1}{2} \times \frac{1}{-1/2} \left| \frac{1}{\xi^{1/2}} \right|_{y^2}^{L^2+y^2} = -\frac{ky}{4\pi\varepsilon_0} \left(\frac{1}{\left(L^2 + y^2\right)^{1/2}} - \frac{1}{y} \right)$$

$$= \frac{k}{4\pi\varepsilon_0} \left(1 - \frac{y}{\left(L^2 + y^2\right)^{1/2}} \right).$$

(c)

Potential at the left end of the rod will be

$$V\left(0,0\right) = \frac{kL}{4\pi\varepsilon_0}.$$

We next find y for which

$$V\left(0, y\right) = \frac{1}{2} V\left(0,0\right) = \frac{kL}{8\pi\varepsilon_0}.$$

Or

$$\frac{k}{4\pi\varepsilon_0}\left(\left(L^2+y^2\right)^{1/2}-y\right)=\frac{kL}{8\pi\varepsilon_0},$$

or

$$\left(L^2+y^2\right)^{1/2}=\frac{L}{2}+y,$$

or

$$L^2+y^2=\frac{L^2}{4}+y^2+Ly,$$

and

$$y=\frac{3}{4}L.$$