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Problem 24.56P (HRW)

A hydrogen atom can be considered as having a central point-like proton of positive charge $+e$ *and an electron of negative charge* $-e$ *that is distributed about the proton according to the volume charge density* $\rho = A\exp(-2r/a_0)$. *A is a constant,* $a_0 = 0.53\times10^{-10}$ m *is the **Bohr radius**, and r is the distance from the centre of the atom.* (a) *We have to find A using the fact that hydrogen is electrically neutral.* (b) *We shall then find the electric field produced by the atom at the **Bohr radius**.*

Solution:

(a)

The volume charge density of the electron cloud is given by the function

$\rho = A\exp(-2r/a_0)$.

Total charge of the electron will be given by the integral

$$\int_0^\infty 4\pi r^2 A e^{-2r/a_0} dr = 4\pi A \int_0^\infty r^2 e^{-2r/a_0} dr .$$

We will calculate the integral

$$\int_0^\infty r^2 e^{-2r/a_0} dr .$$

We make the substitution

$$\frac{2r}{a_0} = \xi.$$

We have

$$\frac{2dr}{a_0} = d\xi.$$

And

$$\int_0^\infty r^2 e^{-2r/a_0} dr = \frac{a_0^3}{8} \int_0^\infty \xi^2 e^{-\xi} d\xi = \frac{a_0^3}{4}.$$

Therefore, the total charge of the electron cloud will be

$$\pi A a_0^3 = -e .$$

Or

$$A = -\frac{e}{\pi a_0^3}.$$

(b)

We next calculate the total charge contained inside a sphere of radius *r* centred on the proton. It will be

$$q(r) = e + \int_0^r 4\pi r^2 A e^{-2r/a_0} dr$$

$$= e + 4\pi A\left(\frac{a_0^3}{4} - \left(\frac{a_0^3}{4} + \frac{a_0^2 r}{2} + \frac{a_0 r^2}{2}\right)e^{-2r/a_0}\right)$$

$$= e - \frac{4e}{a_0^3}\left(\frac{a_0^3}{4} - \left(\frac{a_0^3}{4} + \frac{a_0^2 r}{2} + \frac{a_0 r^2}{2}\right)e^{-2r/a_0}\right)$$

$$= e\left(1 + \frac{2r}{a_0} + \frac{2r^2}{a_0^2}\right)e^{-2r/a_0}.$$

By applying Gauss' law,

$$\varepsilon_0 4\pi r^2 E(r) = q(r),$$

we find that

$$E(r) = \frac{e\left(1 + \frac{2r}{a_0} + \frac{2r^2}{a_0^2}\right)e^{-2r/a_0}}{4\pi\varepsilon_0 r^2}.$$