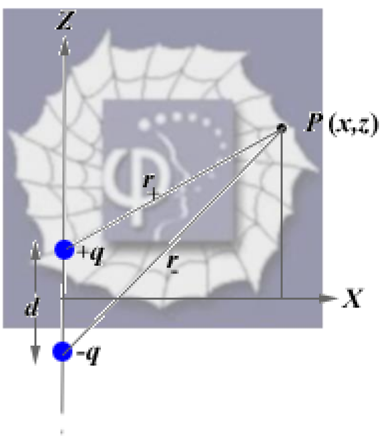
292.

<u>Problem 28.12 (RHK)</u>

We have to show that the components of $\vec{E}$ due to a dipole are given, at distant points, by

$$E_x = \frac{1}{4\pi\varepsilon_0}\frac{3pxz}{\left(x^2+z^2\right)^{5/2}}, \; E_y = \frac{1}{4\pi\varepsilon_0}\frac{p\left(2z^2-x^2\right)}{\left(x^2+z^2\right)^{5/2}},$$

where x and z are coordinates of point P in the figure.



<u>Solution:</u>

We note from the diagram that the distance of the point *P* from the charge +*q* is

$$r_+ = \left(x^2 + \left(z - \tfrac{d}{2}\right)^2\right)^{1/2}$$

and its distance from the charge –*q* is

$$r_- = \left(x^2 + \left(z + \tfrac{d}{2}\right)^2\right)^{1/2}.$$

X-component of the electric field due to the dipole at the point *P* will be

$$E_x = \frac{q}{4\pi\varepsilon_0 r_+^2} \times \frac{x}{r_+} - \frac{q}{4\pi\varepsilon_0 r_-^2} \times \frac{x}{r_-}$$

$$= \frac{qx}{4\pi\varepsilon_0}\left(\frac{1}{r_+^3} - \frac{1}{r_-^3}\right) = \frac{qx}{4\pi\varepsilon_0\left(x^2+z^2\right)^{3/2}}\left(\left(1+\frac{\frac{d^2}{4}-dz}{x^2+z^2}\right)^{-3/2} - \left(1+\frac{\frac{d^2}{4}+dz}{x^2+z^2}\right)^{-3/2}\right)$$

As the point *P* is distant from the dipole, we can assume that

$\frac{\frac{d^2}{4}-dz}{x^2+z^2} = 1$, and also $\frac{\frac{d^2}{4}+dz}{x^2+z^2} = 1$.

We use the approximation that

$\left(1+\xi\right)^n \; ; \; 1+n\xi,$

for $\xi = 1,$

and we get

$$E_x \; ; \; \frac{3qd\,xz}{4\pi\varepsilon_0\left(x^2+z^2\right)^{5/2}} = \frac{3pxz}{4\pi\varepsilon_0\left(x^2+z^2\right)^{5/2}},$$

where we have substituted $qd = p$, for the dipole moment.

The *Z*-component of the electric field is

$$E_z = \frac{q}{4\pi\varepsilon_0 r_+^2} \times \frac{\left(z + \frac{d}{2}\right)}{r_+} - \frac{q}{4\pi\varepsilon_0 r_-^2} \times \frac{\left(z + \frac{d}{2}\right)}{r_-}$$

$$= \frac{q}{4\pi\varepsilon_0} \times \frac{1}{\left(x^2 + z^2\right)^{3/2}} \left(\begin{array}{l} \left(z - \frac{d}{2}\right)\left(1 + \frac{\frac{d^2}{4} - dz}{x^2 + z^2}\right)^{-3/2} - \\ \left(z + \frac{d}{2}\right)\left(1 + \frac{\frac{d^2}{4} + dz}{x^2 + z^2}\right)^{-3/2} \end{array} \right).$$

As the point *P* is distant from the dipole, we can assume that

$\frac{\frac{d^2}{4} - dz}{x^2 + z^2} = 1$, and also $\frac{\frac{d^2}{4} + dz}{x^2 + z^2} = 1$.

We use the approximation that

$\left(1 + \xi\right)^n \; ; \; 1 + n\xi,$

for $\xi = 1,$

and we have

$$E_z \; ; \; \frac{q}{4\pi\varepsilon_0}\times\frac{1}{\left(x^2+z^2\right)^{3/2}}\left(\left(z-\frac{d}{2}\right)\left(1-\frac{\frac{3}{2}\left(\frac{d^2}{4}-dz\right)}{\left(x^2+z^2\right)}\right)-\left(z+\frac{d}{2}\right)\left(1-\frac{\frac{3}{2}\left(\frac{d^2}{4}+dz\right)}{\left(x^2+z^2\right)}\right)\right)$$

On simplifying the algebraic expression, we easily get

$$E_z=\frac{p}{4\pi\varepsilon_0}\times\frac{\left(2z^2-x^2\right)}{\left(x^2+z^2\right)^{5/2}}.$$