77 (c).

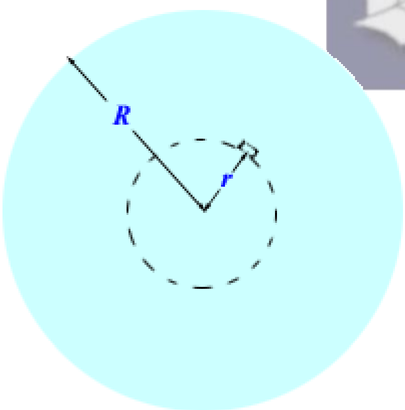
Hydrostatic Equilibrium in Spherical Fluid Mass

We have to show that in a homogeneous spherical fluid mass of radius R and density ρ, pressure at a distance r from the centre of the sphere is

$$p(r)=\frac{2\pi}{3}\rho^2 G\left(R^2-r^2\right).$$

For the Earth taking $R=6.3\times10^6$ m *and* $\rho=5.5\times10^3$ kg m^{-3}, *we have to find the pressure at the centre.*

Solution:



Let us consider a cylindrical surface of unit cross-section width Δr at a distance *r* from the centre of the spherical fluid system in hydrostatic equilibrium. Condition of equilibrium for this fluid element is that the inward gravitational pull on it has to be balanced by the pressure force acting on its opposite faces, each of which is assumed to be unit cross-sectional area. Let the pressure inside the fluid be denoted by the function $p(r)$.

The inward gravitational pull on this fluid element will be the force due to the fluid mass contained in the sphere of radius r and acting from its centre. That is

$$F_g = G\left(\frac{4\pi r^3\rho}{3}\right)\times\left(\frac{\Delta r\rho}{r^2}\right).$$

Net pressure force acting in the outward radial direction will be

$$F_p = p(r) - p(r+\Delta r).$$

Condition of equilibrium

$$F_p = F_g,$$

gives the equation for pressure variation,

$$-\frac{dp}{dr} = \frac{4\pi}{3}\rho^2 Gr.$$

Integrating this equation, we get

$$p(r) = -\frac{2\pi}{3}G\rho r^2 + c.$$

Requiring that at the boundary of the fluid mass pressure is zero, $p(R)=0$, we find

$$c = \frac{2\pi}{3}G\rho^2 R^2.$$

Radial variation of pressure is, therefore, given by the function

$$p(r) = \frac{2\pi}{3} G\rho^2 \left(R^2 - r^2\right).$$

We next calculate the pressure at the centre of the Earth considering it to be a homogeneous spherical fluid mass of radius $R = 6.3 \times 10^6$ m and density $\rho = 5.5 \times 10^3$ kg m^{-3}.

Substituting the values, we find that

$$p_{earth}(0) = 1.68 \times 10^{11} \text{ Pa.}$$