75 (b).

Problem 17.29 (RHK)

A fluid is rotating at constant angular velocity ω *about the central vertical axis of a cylindrical container.*

(a) *We have to show that the variation of pressure in the radial direction is given by*

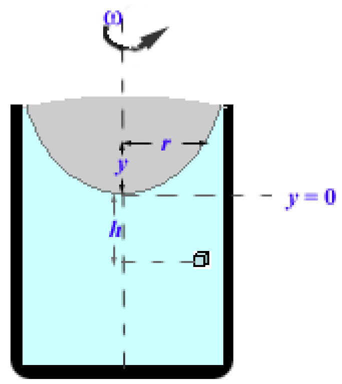
$$\frac{dp}{dr} = \rho\omega^2 r.$$

(b) *By taking* $p = p_c$ *at the axis of rotation* (r=0) *we have to show that the pressure p at any point r is*

$$p = p_c + \tfrac{1}{2}\rho\omega^2 r^2.$$

(c) *We have to show that the liquid surface is of paraboloidal form; that is a vertical cross-section of the surface is the curve* $y = \omega^2 r^2 / 2g$.

(d) *We have to show that the variation of pressure with depth is* $p = \rho gh$.



Solution:

The problem is about the famous Newton's rotating bucket experiment.

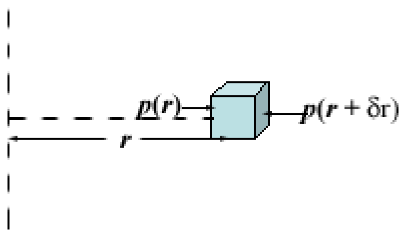
(a)

Consider a fluid mass rotating at constant angular speed ω about the central vertical axis of a cylindrical container. Liquid surface takes the shape as shown in the diagram. What has been shown in the diagram is a vertical cross-section of the rotating fluid. The surface of the fluid is curved. In fact, it is paraboloidal.

We fix the co-ordinate system by measuring vertical distance from the lowest point in the surface of the rotating fluid; it is by symmetry on the central axis.

Fluid below the level $y = 0$ is rotating uniformly with angular speed ω.

We will analyse the dynamics of rotation of the fluid. At a distance *r* from the central axis we consider a fluid element of cross-section ΔA and width δr. Let ρ be the density of the fluid. The mass of this fluid element will be $\rho \Delta A \delta r$. As this fluid element is rotating with angular speed ω at a distance *r* from the rotation axis, it must obtain a centripetal force of magnitude $\rho \Delta A \delta r \omega^2 r$. As the only source of force in the horizontal direction can be fluid pressure, it implies that in a rotating fluid pressure will vary with depth *h* and radial distance *r*. That is the pressure is a function $p(h,r)$. We will set up the equation of motion of the fluid element and from it obtain a differential equation for $p(h,r)$. We fill first obtain the variation of *p* as a function of *r*. As we are considering the rotation of a fluid element at depth *h*, in the following we will suppress the dependence of *p* on *h*.



Centripetal force is provided by the pressure difference at the two opposite faces of the fluid element.

$$p(r+\delta r)\Delta A - p(r)\Delta A = \rho \Delta A \delta r\, \omega^2 r.$$

As $p(r+\delta r) = p(r) + \frac{dp}{dr}\delta r,$ by retaining terms of order in δr, we get

$$\frac{dp}{dr} = \rho\omega^2 r.$$

(b)

It is a linear first order differential equation. Its general solution is

$$p(h,r) = p(h,0) + \tfrac{1}{2}\rho\omega^2 r^2.$$

Calling $p(h,0) = p_c$,

We get,

$$p(h,r) = p_c + \tfrac{1}{2}\rho\omega^2 r^2.$$

On the central axis fluid is at rest as it does not rotate, variation of pressure on the axis with depth is the hydrostatic relation

$$p(h,0) = p_0 + \rho g h,$$

where p_0 is the atmospheric pressure.

(c)

We now will find out how the radial pressure $\frac{1}{2}\rho\omega^2 r^2$ arises.

Radial pressure is due to the curved shape of the liquid surface. Let the height of the liquid surface at radial distance as measured from the level $y = 0$ be y. Then

$$\rho g y = \tfrac{1}{2}\rho\omega^2 r^2 \ ,$$

or

$$y = \frac{\omega^2 r^2}{2g}.$$

It is the equation of a parabola. This curve on rotation about the central axis will trace a paraboloid.