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Problem 3.33 (R)

An excited atom of mass m, initially at rest in frame S, emits a photon and recoils. The internal energy of the atom decreases by ΔE and the energy of the photon is $h\nu$. We have to show that $h\nu = \Delta E\left(1 - \Delta E/2mc^2\right)$.

Solution:

As mass m of the atom is much greater than ΔE, the recoil motion of the atom on emitting a photon will be nonrelativitic. We will use conservation of energy and momentum, and the relativistc relation between energy and momentum for photon in solving this problem.

As the energy of the photon is $h\nu$, its momentum will be $p = h\nu/c$.

As the atom was initially at rest and its momentum was zero, the magnitude of the recoil momentum of the atom will be $p = h\nu/c$. On emission of a photon of energy $h\nu$ the kinetic energy of the atom will be

$$KE = \frac{p^2}{2m} = \frac{\left(h\nu\right)^2}{2mc^2}.$$

We have used the approximation that the change in the rest mass of the atom after emission of a photon will be negligible with respect to its mass m. From conservation of energy we get

$$\Delta E = h\nu + \frac{\left(h\nu\right)^2}{2mc^2},$$

It is a quadratic equation. Its roots are

$$h\nu = -mc^2\ \mathrm{m} \left(1 + \frac{2\Delta E}{mc^2}\right)^{1/2}.$$

As the energy of the photon cannot be negative, the physical root is

$$h\nu = -mc^2 + mc^2\left(1 + \frac{2\Delta E}{mc^2}\right)^{1/2}.$$

As $2\Delta E/mc^2 = 1$, on expanding the square root as a binomial expansion, to leading order we get

$$h\nu = -mc^2 + mc^2\left(1 + \frac{\Delta E}{mc^2} - \frac{\left(\Delta E\right)^2}{2m^2c^4}\right).$$

Therefore,

$$h\nu = \Delta E\left(1 - \Delta E/2mc^2\right).$$