176.

<u>Problem 21.56 (RHK)</u>

A proton, mass m, accelerated in a proton synchrotron to a kinetic energy K strikes a second (target) proton at rest in the laboratory. The collision is entirely inelastic in that the rest energy of the two protons, plus all the kinetic energy consistent with the law of conservation of momentum, is available to generate new particles and to endow them with kinetic energy. We have to show that the energy available for this purpose is given by

$$E_{new} = 2mc^2 \sqrt{\left(1 + \left(\frac{K}{2mc^2}\right)\right)} \ .$$

(b) *We have to find the available energy when* 100-GeV *protons are used in this fashion.* (c) We have to calculate the proton energy for making 100 GeV *available.*

Solution:

A proton, mass m, accelerated in a proton synchrotron to kinetic energy K strikes a second (target) proton at rest in the laboratory. If we assume that the collision is entirely inelastic in that the rest energy of the two protons, plus all the kinetic energy, consistent with the law of conservation of momentum, is available, then this energy can be conveniently found by going to the centre of mass (cm) frame.

Let v be the velocity of the incident proton in the laboratory frame. In the centre of mass frame the incident proton and the target proton will be moving toward each other with the same speed v_{cm}. In the laboratory frame the target proton is at rest and in the cm frame its speed is v_{cm}, the speed of the cm frame with respect to the laboratory frame will be v_{cm}. The speed of the incident proton in the cm frame will be v_{cm}. We use the relativistic velocity addition theorem to find the v_{cm}.

$$v = \frac{v_{cm} + v_{cm}}{1 + v_{cm}{}^2 / c^2} \ ,$$

or

$$\frac{v_{cm}{}^2}{c^2} - \frac{2c}{v} \times \frac{v_{cm}}{c} + 1 = 0.$$

Roots of this quadratic equation are

$$\frac{v_{cm}}{c} = \frac{c}{v}\left(1 \pm \sqrt{1 - v^2/c^2}\right).$$

As $\frac{v_{cm}}{c} < 1,$ the physical solution is

$$\frac{v_{cm}}{c} = \frac{c}{v}\left(1 - \sqrt{1 - v^2/c^2}\right).$$

Total energy of the incident proton and the target proton in the cm frame will be the available energy for particle production. It is

$$E_{new} = \frac{2mc^2}{\sqrt{1 - v_{cm}{}^2 / c^2}} .$$

Substituting $\frac{v_{cm}}{c} = \frac{c}{v}\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right)$ and carrying out algebraic simplifications, we find

$$E_{new} = \sqrt{2}\, mc^2 \left(1 + \frac{1}{\sqrt{1 - v^2/c^2}} \right).$$

Using $\frac{K}{mc^2} + 1 = \frac{1}{\sqrt{1 - v^2/c^2}}$, expression for E_{new} can be expressed in the form

$$E_{new} = 2\, mc^2 \left(1 + \frac{K}{2mc^2} \right)^{1/2}.$$

(b)

From the above result we calculate the available energy when 100-GeV protons are used. It will be

$$E_{new} = 2 \times .938 \left(1 + \frac{100}{2 \times .938} \right)^{1/2} \text{ GeV} = 13.8 \text{ GeV}.$$

(c)

Proton energy required to make 100 GeV available can be found using the relation derived above. It is given by the expression

$$\sqrt{1 + \frac{K}{2mc^2}} = \frac{100}{2 \times .938} = 53.3.$$

This gives

$$\frac{K}{2mc^2} = 53.3^2 - 1 = 2840.4,$$

or

$$K = 2840.4 \times (2 \times .938)\text{ GeV} = 5329\text{ GeV}.$$