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Problem 19.44 (RHK)

We consider a standing wave that is the sum of two waves travelling in opposite directions but otherwise identical. We have to show that the maximum kinetic energy in each loop of the standing wave is $2\pi^2 \mu y_m{}^2 fv$.

Solution:

Let us represent the two identical sinusoidal waves travelling in opposite directions by the functions

$y_1 = y_m \sin(kx - \omega t)$,

and

$y_2 = y_m \sin(kx + \omega t)$.

Standing wave formed by the superposition of these waves is described by the function

$$y = y_1 + y_2 = (2y_m \sin kx)\cos \omega t.$$

Speed of vibration at position x of the string is given by the time derivative of $y(x,t)$. That is

$$v(x,t) = \frac{dy}{dt} = -2y_m \omega \sin kx \sin \omega t.$$

At a given *x*, the maximum speed of oscillation is

$$\left(v(x)\right)_{\max} = 2y_m\left(2\pi f\right)\sin kx.$$

We have expressed ω in terms of *f*, $\omega = 2\pi f$.

End points of the first loop are $x = 0$ and $x = \lambda/2$.

Therefore, the maximum kinetic energy in each loop can be calculated by integrating maximum KE of element of mass μdx at position *x* over the length of the loop. We have

$$\left(\text{KE of one loop}\right)_{\max} = \frac{1}{2}\int_0^{\lambda/2} \mu dx\left(4\pi^2 f^2\right)4y_m{}^2\sin^2 kx$$

$$= 8\pi^2 f^2 y_m{}^2 \mu \int_0^{\lambda/2} dx \sin^2\left(\frac{2\pi x}{\lambda}\right)$$

$$= 4\pi^2 f^2 y_m{}^2 \mu \int_0^{\lambda/2} dx\left(1 - \cos\left(\frac{4\pi x}{\lambda}\right)\right)$$

$$= 2\pi^2 f^2 y_m{}^2 \mu\lambda = 2\pi^2\left(fv\right)y_m{}^2\mu.$$